VARIATIONAL INEQUALITIES WITH LOWER DIMENSIONAL OBSTACLES*

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ABSTRACT

We study a two-dimensional variational inequality where the interior constraining function, or obstacle, is defined only on a segment of the domain. A simple proof of the existence and smoothness of the solution to this type of problem is available via the results of a recent paper by Lewy and Stampacchia. Important special cases include the Dirichlet Integral and the area integrand.

1. In this paper we study a two-dimensional variational inequality where the interior constraining function, or obstacle, is defined only on a segment of the domain. We show how a recent theorem of Lewy and Stampacchia [7] may be used to derive a simple proof of the existence and smoothness of the solution to this type of problem.

Let Ω be a strictly convex domain in the $x = (x_1, x_2)$ plane, σ a closed straight segment in Ω assumed to lie on the x_1-axis, and $f \in C^{1,\alpha}(\sigma)$ a non-negative function assumed to vanish at the endpoints of σ. Suppose that $a(p)$ is a locally coercive C^1 vector field on R^2. Consider the convex set $\mathscr{K}$ of Lipschitz functions in Ω which are constrained to lie above f on σ and to vanish on $\partial\Omega$. We wish to prove the existence of a $u \in \mathscr{K}$ such that

$$\int_\Omega a_j(u_x)(v-u)_{x_j}\,dx \geqq 0 \text{ for all } v \in \mathscr{K}.$$

We establish these properties under an assumption about the regularity of the solution to the Dirichlet problem for the equation $-\partial/\partial x_j\,(a_j(u_x)) = 0$ in domains which are convex, but not strictly convex.

One case of this problem arises in minimal surfaces and has been treated by Nitsche [9]. We obtain a generalization of his result which, in particular, requires

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no assumptions of symmetry in the configuration of σ and Ω or concavity of f. Another case arises from the Dirichlet Integral. This has been discussed by Lewy ([5], [6]). Our results here do not yield anything new.

On the other hand, the results of this paper may be applied to a wide class of variational inequalities obtained from locally coercive vector fields which are elliptic in a suitable sense. We describe this situation below. Although our concern has been with two dimensional problems, many of these techniques also apply to higher dimensions.

We establish that the solution to the problem of Lewy and Stampacchia (§5, [7]) for a suitable Lipschitz obstacle is actually the solution to the problem here for the constraint f. To construct a suitable obstacle, we first assume that $f' = 0$ at the endpoints of σ. This restriction is then removed. If f is assumed to be concave, it follows by an argument of Nitsche [9] that the subset of σ where the solution $u = f$ is a connected segment. For situations involving minimal surfaces, it then follows that $u_{x_1} \in C^0(\overline{\Omega})$ and u_{x_2} is continuous on one-sided approach to σ.

Here are the precise assumptions about $a(p)$. Let $a(p) = (a_1(p), a_2(p)), p=(p_1,p_2)$, be a vector field of class C^1 which is locally coercive in the sense that given a compact set $C \subset R^2$, there is a $\nu = \nu(C) > 0$ satisfying

$$(a(p) - a(q)) \cdot (p - q) \geqq \nu(C) \left| p - q \right|^2 \textit{ for } p, q \in C. \tag{1.1}$$

We also assume the condition below:

(1.2) *Let G be any convex, but not necessarily strictly convex, domain in the x-plane with C^2 boundary and let $\phi(x) \in C^{1,\alpha}(\partial G)$, for some α, $0 < \alpha < 1$. Then there exists a $w \in C^{0,1}(\bar{G})$ such that*

$$\frac{\partial}{\partial x_j} a_j(w_x) = 0 \textit{ in } G$$

$$w = \phi \textit{ on } \partial G.$$

For the discussion here, it is important that w be Lipschitz in $\bar{G}$. One instance of (1.2) is obtained by setting $a_j(p)=p_j/W$, $W=\sqrt{1+p^2}$, which is the monotone vector field determined by the non-parametric area integrand. The reader is invited to refer to the appendix for a discussion of this point.

Another instance of (1.2) is obtained when the vector field $a(p)$ is elliptic in a suitable sense. For $a(p) = (a_1(p), a_2(p))$, a two-dimensional situation, it suffices to assume that the ratio of the eigenvalues of the symmetric part of $(\partial a_j(p)/\partial p_k)$ is bounded above and below. Namely, given $a(p)$ let $\lambda_{\max} \geqq \lambda_{\min} \geqq 0$ denote the eigenvalues of the matrix

$$a_{jk} = \tfrac{1}{2}\left(\frac{\partial a_j}{\partial p_k} + \frac{\partial a_k}{\partial p_j}\right).$$

Then condition (1.2) is satisfied if $1 \leqq \lambda_{\max}/\lambda_{\min} \leqq c$, c independent of $p \in R^2$. This conclusion results because the estimates of Bers and Nirenberg (cf. [1], p. 263) depend only on the size of the coefficients and the modulus of ellipticity of an equation, but not their continuity properties. One special case of this situation is the vector field $a_j(p) = (1 + p^2)^{t/2-1} p_j$, $1 < t \leqq 2$, which is locally coercive but not coercive, and has been considered by Brezis and Stampacchia ([2]).

Denote by A the operator defined by the pairing

$$(Au, v) = \int_\Omega a_j(u_x) v_{x_j} dx, \quad u, \ v \in H_0^1(\Omega)$$

whenever the right hand side exists. If $u \in C^{0,1}(B)$ and $(Au, v) = 0$ for all $v \in H_0^1(B)$, B an open ball in Ω, then $u \in C^{1,\lambda}(B_0) \cap H^{2,q}(B_0)$ for any sub-ball B_0 with $\bar{B}_0 \subset B$ and u is a solution to

$$Au = -\frac{\partial}{\partial x_j}(a_j(u_x)) = 0 \text{ a.e. in } B.$$

If ψ is any function defined in Ω, we set $\mathscr{K}_\psi = \{v \in C_0^{0,1}(\bar{\Omega}) : v \geqq \psi \text{ in } \Omega\}$.

2. The solution to a variational inequality with a Lipschitz obstacle

Let $\psi(x) \in C^{0,1}(\bar{\Omega})$ satisfy $\psi < 0$ on $\partial\Omega$. By [7] (Theorem 2.2 and §5), there is a unique $u \in \mathscr{K}_\psi$ satisfying

$$(2.1) \qquad (Au, v - u) \geqq 0 \text{ for all } v \in \mathscr{K}_\psi.$$

If $\psi \in C^2(\bar{\Omega})$, then $u \in C^{1,\lambda}(\bar{\Omega}) \cap H^{2,q}(\bar{\Omega})$ for each λ, $0 < \lambda < 1$, and q, $1 < q < \infty$. The first step towards attaining the solution to (2.1) when ψ is Lipschitz is the approximation of the given obstacle by smooth obstacles. The solution is proven to be the limit of the solutions to the problems for smooth obstacles. Suppose that in a given ball $B \subset \Omega$, $\psi \in H^{2,q}(B)$. It follows that in any sub-ball $B_0 \subset B$, the solution u of (2.1) is in $H^{2,q}(B_0)$. The proof of this assertion is immediate from the local quality of the well-known estimates for elliptic equations ([10] and [8] for the linear case).

One also employs that $\psi \in C^2(\Omega)$ implies

$$\int_{\Omega}\{a_j(u_x)\zeta_{x_j} - \max(A\psi, 0)F(x)\}dx = 0 \text{ for } \zeta \in H_0^1(\Omega),$$

where F is a certain function with $\| F \|_{L^\infty(\Omega)} \leqq 1$, ([7], equation 4.6).

We point out that a strict maximum principle is valid for the difference of two solutions to the homogeneous equation $Au = 0$. We check this. Let G be a domain in R^2, and let u, $v \in H^{2,q}(G) \cap C^0(\bar{G})$ satisfy

$$\int_G a_j(u_x)\zeta_{x_j}dx = 0 \quad \text{and} \int_G a_j(v_x)\zeta_{x_j}dx = 0, \ \zeta \in H_0^1(G)$$

and

$$u \geqq v \text{ on } \partial G.$$

By setting $w = \max(u, v)$ it is easy to see that $w = u$ in G, so that $u \geqq v$ in G. Since $a_j(p) \in C^1$ and $u_x, v_x \in C^{0,\lambda}(G)$, $\lambda = 1 - 2/q$, we may write

$$a_j(u_x) - a_j(v_x) = b_{1j}(u - v)_{x_1} + b_{2j}(u - v)_{x_2},$$

where

$$b_{kj}(x) = \frac{\partial a_j}{\partial p_k}(v_x + t_j(x)(u_x - v_x)), \ 0 \leqq t_j(x) \leqq 1.$$

At any $x^0 \in G$ where $u(x^0) = v(x^0)$, $u - v$ has a minimum so that $u_x(x^0) = v_x(x^0)$. By the coerciveness of $a_j(p)$,

$$(*) \qquad b_{kj}(x)\xi_j\xi_k > C|\xi|^2 \text{ for all } \xi = (\xi_1, \xi_2)$$

and $x = x^0$. Since u_x, v_x are continuous and $|t_j| < 1$ there is a ball $B = \{|x - x^0| < \varepsilon\}$ where (*) holds for all $x \in \bar{B}$. Hence, $w = u - v$ is the solution to

$$\int_B b_{kj}w_{x_k}\zeta_{x_j}dx = 0 \quad \text{for } \zeta \in H_0^1(B)$$

Now $w \in H^{2,q}(B)$ so that the well known maximum principle ([1] p. 262) implies that if $w(x^0) = \inf_B w(x)$ for some $x^0 \in B$, then $w(x) = w(x^0)$ for all $x \in B$.

LEMMA 1. *Let $\psi(x)$ be a Lipschitz obstacle in Ω. Suppose that $\psi \in H^{2,q}(G)$ and $A\psi = 0$ in D, for some connected open subset $D \subset \Omega$ where $\psi \geqq 0$ and any ball $G \subset D$. Let $u \in \mathscr{K}_\psi$ satisfy (2.1). Then either $u(x) = \psi(x)$ for all $x \in D$ or $u(x) > \psi(x)$ for all $x \in D$.*

The proof of the lemma is evident.

3. A Lipschitz obstacle

We construct a Lipschitz obstacle under the hypothesis that $f = f' = 0$ at the endpoints of σ and $f \in C^{1,\alpha}(\sigma)$. Let γ_+ be a C^2 strictly convex arc in $x_2 \geqq 0$ which connects the endpoints of σ in such a way that $\sigma \cup \gamma_+$ is a C^2 curve and $(\sigma \cup \gamma_+) \cap \{x_2 = 0\} = \sigma$. Let D^+ denote the domain enclosed by the convex curve $\sigma \cup \gamma_+$ and determine ψ_+ by

$$\frac{\partial}{\partial x_j} a_j(\psi_+(x)) = 0 \text{ in } D^+$$

$$\psi_+(x) = f \text{ on } \sigma$$

$$\psi_+(x) = 0 \text{ on } \gamma_+$$

The hypotheses about $a_j(p)$ imply that $\psi_+ \in C^{0,1}(D^+) \cap H^{2,q}_{\mathrm{loc}}(D^+)$. Similarly, in the reflection of D^+, $D^- = \{x : \bar{x} = (x_1, -x_2) \in D^+\}$, determine ψ_- by

$$\frac{\partial}{\partial x_j} a_j(\psi_-(x)) = 0 \text{ in } D^-$$

$$\psi_-(x) = f \text{ on } \sigma$$

$$\psi_-(x) = 0 \text{ on } \gamma_-, \text{ the reflection of } \gamma_+.$$

We define $\psi(x)$ by $\psi(x) = \psi_+(x)$ for $x \in \bar{D}^+$, $\psi(x) = \psi_-(x)$ for $x \in \bar{D}^-$, and extend this ψ to a Lipschitz function, still denoted by ψ, so that $\psi \leqq 0$ in $\Omega - (\bar{D} \cap \bar{D}^-)$ and $\psi < 0$ on $\partial\Omega$.

There is a unique $u \in \mathscr{K}_\psi$ such that $(Au, v - u) \geqq 0$ for all $v \in \mathscr{K}$. It is not difficult to see that u is not identical to ψ where ψ is positive. Let us demonstrate the useful fact.

Lemma 2. *Let u be the solution to the variational inequality $u \in \mathscr{K}_\psi : (Au, v-u) \geqq 0$ for all $v \in \mathscr{K}_\psi$, where ψ is the obstacle constructed from f discussed in this section. Then $u > \psi$ in $D^+ \cup D^-$.*

Proof. There are a number of ways to prove this lemma. We employ one which gives a simple proof that a Lipschitz super-solution of A has no interior minimum without reducing to a constant (cf. [11] for example).

Let us note first that u is a super-solution to A, which means that $\zeta \in H^1_0(\Omega)$ and $\zeta \geqq 0$ implies that $(Au, \zeta) \geqq 0$. For if $\zeta \in H^1_0$, then $u + \zeta$ is the limit of functions in $\mathscr{K}$ so that $(Au, \zeta) \geqq 0$.

If $u(x) = \psi(x)$ for some $x \in D^+$, then $u(x) = \psi(x)$ for all $x \in D^+$ by Lemma 1, and in particular assumes the interior minimum value zero on γ_+. Let $x_0 \in \gamma_+$ and $B = \{|x - x^0| < \varepsilon\}$ be a ball in Ω which does not intersect the closed set $\bar{D}^+ \cup \partial\Omega$. We determine $w(x) \in H^1(B) \cap H^{2,q}_{\mathrm{loc}}(B)$ as the unique solution to

$$(3.1)\qquad \begin{aligned} -\frac{\partial}{\partial x_j} a_j(w_x) &= 0 \text{ in } B \\ w &= u \text{ on } \partial B \end{aligned}$$

For the moment we assume the existence of w, a point to which we shall return. Then $w \in C^{1,\lambda}(B)$, $0 < \lambda < 1$. With $\zeta = \max(u,w) - u \geqq 0$ in B and in $H_0^1(B)$, $0 \leqq \int_{\{w>u\}}(a_j(w_x) - a_j(u_x))(w-u)_{x_j}dx = \int_B (a_j(w_x) - a_j(u_x))\zeta_{x_j}dx$ by coerciveness of A. On the other hand, since w is a solution and u a super-solution of A,

$$\int_B (a_j(w_x) - a_j(u_x))\zeta_{x_j}dx = -\int_B a_j(u_x)\zeta_{x_j}dx \leqq 0.$$

We conclude, since w, $u \in C^0(B)$, that $\max_B(w,u) = u$. Now $w(x) > \inf_{\partial B} w(y) = 0$ for $x \in B$, hence $u(x^0) \geqq w(x^0) > 0$, which contradicts that $u(x^0) = \psi(x^0) = 0$.

We now discuss briefly the existence of w, the solution to (3.1). The difficulty encountered by the application of direct methods is that $a_j(p)$ is not coercive. On the other hand the boundary data is Lipschitz on a strictly convex set. This enables us, using an idea of Radó, to conclude the existence of a solution to (3.1) which is Lipschitz in $\bar{B}$ and has the same Lipschitz constant as $u(x)$. This type of problem has been treated by Stampacchia and Hartman [3].

4. A special case

THEOREM 1. *Let $a(p)$ be a C^1 vector field satisfying* (1.1) *and* (1.2). *Let $f \in C^{1,\alpha}(\sigma)$ be a non-negative function such that $f = f' = 0$ at the endpoints of σ. Then there exists a unique $u \in \mathscr{K}_f$ such that*

$$(4.1)\qquad (Au, v-u) \geqq 0 \text{ for all } v \in \mathscr{K}_f.$$

Let $B \subset \bar{B} \subset \{x \in \Omega : u(x) > f(x)\}$ be any ball. Then $u \in H^{2,q}(B)$ and satisfies

$$-\frac{\partial}{\partial x_j} a_j(u_x) = 0 \quad \text{a.e. in } \{x \in \Omega : u(x) > f(x)\}.$$

In the statement of the Theorem, $\mathscr{K}_f = \{v \in C^{0,1}(\Omega) : v \geqq f \text{ on } \sigma \text{ and } v = 0 \text{ on } \partial\Omega\}$.

To demonstrate this theorem, we shall show that the solution of (2.1) for the Lipschitz obstacle ψ of §3 satisfies (4.1). We note that by Lemmas 1 and 2, $\{x \in \Omega : u = \psi\} \subset \sigma$ for the solution of (2.1).

LEMMA 3. *Let $v \in \mathscr{K}_f$. Then $(Au, v-u) \geqq 0$.*

PROOF. Let $\varepsilon > 0$ and define $w_\varepsilon(x) = \max(\psi(x), v(x) + \varepsilon)$, where $v \in \mathscr{K}_f$ is arbitrary. Since $v + \varepsilon > f$ on σ and $v \geqq 0$ on $\partial\Omega$, we conclude that $\{x \in \Omega : \psi(x) \geqq v(x) + \varepsilon\} \subset D^+ \cup D^-$. Hence the Lipschitz function

$$\zeta(x) = \begin{cases} \psi(x) - (v(x) + \varepsilon) & \text{where } \psi(x) > v(x) + \varepsilon \\ 0 & \text{elsewhere} \end{cases}$$

has its support in the set $\{x : u(x) > \psi(x)\}$. Now,

$$\int_\Omega a_j(u_x)(w_\varepsilon - u)_{x_j} dx = \int_{\{\psi \leqq v+\varepsilon\}} a_j(u_x)(v - u)_{x_j} dx + \int_{\{v+\varepsilon<\psi\}} a_j(u_x)(\psi - u)_{x_j} dx$$

$$= \int_\Omega a_j(u_x)(v - u)_{x_j} dx + \int_{\{v+\varepsilon<\psi\}} a_j(u_x)(\psi - u)_{x_j} dx - \int_{\{v+\varepsilon<\psi\}} a_j(u_x)(v - u)_{x_j} dx$$

$$= \int_\Omega a_j(u_x)(v - u)_{x_j} dx + \int_{\{v+\varepsilon<\psi\}} a_j(u_x)\zeta_{x_j} dx.$$

The second integral vanishes in view of the nature of the support of ζ. Hence for each $\varepsilon > 0$,

$$\int_\Omega a_j(u_x)(v - u)_{x_j} dx = \int_\Omega a_j(u_x)(w_\varepsilon - u)_{x_j} dx.$$

It is very easy to check that $w_\varepsilon \to w_0$ in $H_0^1(\Omega)$.

Hence, since $w_0 \in \mathscr{K}_\psi$,

$$\int_\Omega a_j(u_x)(v - u)_{x_j} = \int_\Omega a_j(u_x)(w_0 - u)_{x_j} dx \geqq 0.$$

Q.E.D.

5. The general case

We now remove the restriction that $f' = 0$ at the endpoints of σ. This is a simple procedure whose underlying conception is that the solution u should exceed the constraint f at the endpoints of σ. This would suggest that the behavior of f near the endpoints of σ does not influence u.

LEMMA 4. *Let $h, g \in C^{1,\alpha}(\sigma), \sigma = [\alpha_1, \alpha_2]$, satisfy $h(\alpha_j) = h'(\alpha_j) = g(\alpha_j) = g'(\alpha_j) = 0$, $j = 1, 2$, and suppose that $0 \leqq g(x) \leqq h(x)$ for $\alpha_1 < x_1 < \alpha_2$. Let V and u denote the solutions of* (4.1) *for g and h respectively. Then $V(x) \leqq u(x)$ in Ω.*

PROOF. The demonstration amounts to characterizing V as the infimum of functions $v \in \mathscr{K}_g$ which are super-solutions of A. Namely, v is called a super-solution of A if

$$\int_\Omega a_j(v_x)\zeta_{x_j} dx \geqq 0 \text{ for all } \zeta \in C_0^{0,1}(\overline{\Omega}), \quad \zeta \geqq 0.$$

We refer to ([7] §6).

THEOREM 2. *Let $a_j(p)$ be a C^1 vector field satsifying* (1.1), (1.2). *Let $f \in C^{1,\alpha}(\sigma)$ be nonnegative and vanish at the endepoints of σ. Then there exists a unique $u \in \mathscr{K}_f$ such that*

$$\int_\Omega a_j(u_x)(v-u)_{x_j}dx \geqq 0 \text{ for all } v \in \mathscr{K}_f. \tag{5.1}$$

Let $B \subset \bar{B} \subset \{x \in \Omega: u(x) > f(x)\}$ be any ball. Then $u \in H^{2,q}(B)$ and

$$-\frac{\partial}{\partial x_j}a_j(u_x) = 0 \text{ a.e. in } \{x \in \Omega: u(x) > f(x)\}. \tag{5.2}$$

PROOF. Let $\sigma = [\alpha_1, \alpha_2]$ and select a proper subinterval $\sigma_0 = [\beta_1, \beta_2]$ of σ. Choose $g(x) \in C^{1,\alpha}(\sigma_0)$ with $g(\beta_j) = g'(\beta_j) = 0$, $j = 1,2$, such that $g(x) \leqq f(x)$ in σ_0. Let $V(x)$ denote the solution of (5.1) for the constraint $g(x)$. Near α_j, V is a solution to the homogeneous equation (5.2), hence, $V(\alpha_j) > 0 = f(\alpha_j)$. Therefore, there are points γ_1 and γ_2 interior to σ and not in σ_0 such that $\gamma_1 = \inf\{x_1: V(x)f(x) = 0\}$ and $\gamma_2 = \sup\{x_1: V(x) - f(x) = 0\}$.

We choose $h(x) \in C^{1,\alpha}(\sigma)$. such that $h(\alpha_j) = h'(\alpha_j) = 0$ and

$$h = f \text{ for } \gamma_1 \leqq x_1 \leqq \gamma_2$$
$$h \leqq f \text{ for } \alpha_1 \leqq x_1 \leqq \gamma_1 \text{ and } \gamma_2 \leqq x_1 \leqq \alpha_2.$$

Such a choice of h exists. Let $u(x)$ be the solution of (5.1) for $h(x)$. By our construction, $g(x) \leqq h(x)$ in σ; whence, $V(x) \leqq u(x)$ in Ω by Lemma 4. In particular, for $x_1 \in [\alpha_1, \gamma_1] \cup [\gamma_2, \alpha_2]$, $u(x) \geqq V(x) \geqq f(x)$. Hence $u \in \mathscr{K}_f$. Since $h \leqq f$, $\mathscr{K}_f \subset \mathscr{K}_h$. Therefore, if $v \in \mathscr{K}_f$, $(Au, v-u) \geqq 0$. Q.E.D.

We draw some additional conclusions. If f is concave, then Nitsche's result ([9], p. 96) may be applied to conclude that the set $\tau = \{x: u(x) = f(x)\}$ is a connected subinterval of σ. The proof in [9] is for the case of the nonparametric area integrand, but it may be generalized to an arbitrary $a(p) \in C^1$ which is locally coercive. However, let us apply [9] directly and [4] to obtain

COROLLARY 1. *Let $f \in C^{1,\alpha}(\sigma)$ be convex. Then there is a unique $u \in \mathscr{K}_f$ satisfying*

$$\int_\Omega \sqrt{1 + u_{x_1}^2 + u_{x_2}^2}\, dx_1 dx_2 = \min_{v \in \mathscr{K}_f} \int_\Omega \sqrt{1 + v_{x_1}^2 + v_{x_2}^2}\, dx_1 dx_2.$$

The set $\tau = \{x: u(x) = f(x)\}$ is a connected subiterval of σ and $u_{x_1} \in C^0(\bar{\Omega})$ and u_{x_2} is continuous on one-sided approach to τ.

It is possible to extend Theorem 2 to the case where f is merely continuous by carrying through the program of [7]. In those cases where there is a Harnack inequality for $a(p)$, the second conclusion, (5.2), of Theorem 2 remains valid.

Appendix

In this section we describe the verification of the hypothesis (1.1) for the minimal surface equation by means of the boundary regularity theorem for parametric minimal surfaces ([13], [14], [16]). Let G be any convex, but not necessarily strictly convex, domain in the x-plane with smooth boundary ∂G. Let $\phi \in C^{1,\alpha}(\partial G)$. In fact, we could allow $\phi \in C^1(\partial G)$ with ϕ' satisfying a Dini condition, in view of Warschawski's result [15].

By a result of Radó ([15]) the Dirichlet problem

$$-\frac{\partial}{\partial x_j}\left(\frac{u_{x_j}}{W}\right) = 0 \text{ in } G$$
$$u = \phi \text{ on } \partial G$$

possesses a unique solution, continuous in $\bar{G}$ (of also [12]). Let us suppose that $x = 0 \in \partial G$, that the tangent to ∂G at 0 is $(1,0)$, and that G lies in the half plane $\{x_2 \geqq 0\}$.

For the minimal surface $S = \{(x, x_3)\colon x_3 = u(x),\ x \in G\}$, bounded by a Jordan curve, we choose a conformal representation $X(\zeta) = (x_1(\zeta), x_2(\zeta), x_3(\zeta))$ in $\mathfrak{Im}\,\zeta > 0$, $\zeta = \xi + i\eta$. Suppose $X(0) = (0, 0, \phi(0))$. The vector $X(\zeta)$ is harmonic and satisfies the isothermal relations $X_\xi(\zeta)^2 = X_\eta(\zeta)^2$ and $X_\xi(\zeta)\cdot X_\eta(\eta) = 0$ in $\mathfrak{Im}\,\zeta > 0$. According to the parametric regularity theorem, $X(\xi) \in C^{1,\alpha}(\mathfrak{Im}\,\zeta \geqq 0)$.

We set $f_j(\zeta) = x_{j\xi}(\zeta) - i x_{j\eta}(\zeta)$, $1 \leqq j \leqq 3$. Hence

(A.1) $$f_j(\zeta) = c_j + O(\zeta^\alpha),\ 1 \leqq j \leqq 3, \text{ for } |\zeta| \text{ small and } \mathfrak{Im}\,\zeta \geqq 0,$$

and at $\zeta = 0$, the isothermal relations take the form

(A.2) $$c_1^2 + c_2^2 + c_3^2 = 0.$$

Now we note:

i) The convexity of G implies that the harmonic $x_2(\zeta)$ has a minimum at $\zeta = 0$; hence, $\mathfrak{Im}\,c_2 = -x_{2\eta}(0) < 0$.

On the other hand, since $x_2(\xi)$ has a minimum at $\xi = 0$, $\mathrm{Re}\, c_2 = x_{2\xi}(0) = 0$. Hence $c_2 \neq 0$ is imaginary.

ii) Since $x_3(\xi) = \phi(x_1(\xi))$, $\mathrm{Re}\, c_3 = \phi'(0)\ \mathrm{Re}\, c_1$. By (A.2),

$$(1 + \phi'(0)^2)(\mathrm{Re}\, c_1)^2 = (\mathfrak{Im}\,c_1)^2 + |c_2|^2 + (\mathfrak{Im}\,c_3)^2 > 0,$$

so that $\mathrm{Re}\, c_1 \neq 0$.

iii) The Jacobian of the mapping from ζ to x,

$J = \mathfrak{Im} f_1(\zeta)\overline{f_2(\zeta)} = -i(\mathrm{Re} c_1)c_2 + O(\zeta^\alpha)$, is continuous and not zero at $\zeta = 0$. It follows by a direct computation that $u_{x_1}(x)$ and $u_{x_2}(x)$ are continuous at $x=0$.

Added in proof. After this paper was submitted, Prof. E. Giusti informed the author that there is some overlap between this work and [2a]. The author would like to thank Prof. Giusti for several discussions of this subject. The problem treated in [2a] concerns the area integrand in $n \geqq 2$ dimensions and obstacles defined on lower dimensional submanifolds. Its intersection with the present work is the case $a_j(p) = p_j/W$ and f the restriction to σ of a $C^2(\bar{\Omega})$ function. (October 11, 1971)

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